

Exponential propagation for fractional reaction-diffusion cooperative systems with fast decaying initial conditions.

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Abstract

We study the time asymptotic propagation of sectorial solutions to the fractional reaction-diffusion cooperative systems. We prove that the propagation speed is exponential in time, and we find the precise exponent of propagation. This exponent depends on the smallest index of the fractional laplacians and on the principal eigenvalue of the matrix $DF(0)$ where F is the reaction term. We also note that this speed does not depend on the space direction.

1 Introduction

The reaction diffusion equation with Fisher-KPP nonlinearity

$$\partial_t u + (-\Delta)^\alpha u = f(u) \quad (1.1)$$

with $\alpha = 1$, has been the subject of intense research since the seminal work by Kolmogorov, Petrovskii, and Piskunov [1]. Of particular interest are the results of Aronson and Weinberger [2] which describe the evolution of compactly supported data. They showed that there exists a critical threshold $c^* = 2\sqrt{f'(0)}$ such that, for any compactly supported initial value u_0 in $[0, 1]$, if $c > c^*$ then $u(t, x) \rightarrow 0$ uniformly in $\{|x| \geq ct\}$ as $t \rightarrow +\infty$ and if $c < c^*$ then $u(t, x) \rightarrow 1$ uniformly in $\{|x| \leq ct\}$ as $t \rightarrow +\infty$. This corresponds to a linear propagation of the fronts. In addition, (1.1) admits planar travelling wave solutions connecting 0 and 1.

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Reaction-diffusion equations with fractional Laplacian, that is when $\alpha \in (0, 1)$ in (1.1), appear in physical models when the diffusive phenomena are better described by Lévy processes allowing long jumps, than by Brownian processes - obtained when $\alpha = 1$. The Lévy processes occur widely in physics, chemistry and biology. Recently these models have attracted much interest. In connection with the discussion given above, in the recent paper [3], Cabré and Roquejoffre showed that for any compactly supported initial condition, or more generally for initial values decaying faster than $|x|^{-d-2\alpha}$, where d is the dimension of the spatial variable, the speed of propagation becomes exponential in time. They also showed that no travelling wave exist. Their result was sharpened and extended in [4], who proposed a new (and more flexible) argument to treat models of the form (1.2). In the case in which the initial condition decay slowly, [5] states that the solution spreads exponentially faster with a larger index than in the previous case. All these results are in great contrast with the case $\alpha = 1$. They indeed notice that diffusion only plays a role for small times, the large time dynamics being given by a simple transport equation. The scheme of their proof will be reproduced here, but some steps - and this is why it makes system (1.2) worth studying - become less easy. The small time study will require the manipulation of some Polya integrals, and the transport equation will also become more complex.

The work on the single equation (1.1) can be extended to reaction-diffusion systems. The first definitions of spreading speeds for cooperative systems in population ecology and epidemic theory are due to Lui in [6]. In a series of papers, Lewis, Li and Weinberger [7], [8], [9] studied spreading speeds and travelling waves for a particular class of cooperative reaction-diffusion systems, with standard diffusion. Results on single equations in the singular perturbation framework proved by Evans and Souganidis in [10] have also been extended by Barles, Evans and Souganidis in [11]. The viscosity solutions framework is studied in [12], with a precise study of the Harnack inequality. In these papers, the system under study is of the following form

$$\partial_t u_i - d_i \Delta u_i = f_i(u),$$

where, for $m \in \mathbb{N}^*$, $u = (u_i)_{i=1}^m$ is the unknown.

For all $i \in \llbracket 1, m \rrbracket := \{1, \dots, m\}$, the constants d_i are assumed to be positive as well as the bounded, smooth and Lipschitz initial conditions, defined from \mathbb{R}^d to \mathbb{R}_+ . The essential assumptions concern the reaction term $F = (f_i)_{i=1}^m$. This term is assumed to be smooth, to have only two zeroes 0 and $a \in \mathbb{R}^m$ in $[0, a]$, and for all $i \in \llbracket 1, m \rrbracket$, each f_i is nondecreasing in all its components, with the possible exception of the i th one. The last assumption means that the system is cooperative. Under additional hypotheses, which imply that the point 0 is unstable, the limiting behaviour of the solution $u = (u_i)_{i=1}^m$ is understood.

Here, we focus on similar systems, keeping the same assumptions on f , but considering that at least one diffusive term is given by a fractional Laplacian. More precisely, we focus on the large time behaviour of the solution $u = (u_i)_{i=1}^m$, for $m \in \mathbb{N}^*$, to the fractional reaction-diffusion system:

$$\begin{cases} \partial_t u_i + (-\Delta)^{\alpha_i} u_i &= f_i(u), & t > 0, x \in \mathbb{R}^d, \\ u_i(0, x) &= u_{0i}(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

where

$$\alpha_i \in (0, 1] \quad \text{and} \quad \alpha := \min_{i \in \llbracket 1, m \rrbracket} \alpha_i < 1.$$

Note that when $\alpha_i = 1$, then $(-\Delta)^{\alpha_i} = -\Delta$. As general assumptions, we impose, for all $i \in \llbracket 1, m \rrbracket$, the initial condition u_{0i} to be nonnegative, non identically equal to 0, continuous and to satisfy

$$u_{0i}(x) = O(|x|^{-(d+2\alpha_i)}) \quad \text{as} \quad |x| \rightarrow +\infty. \quad (1.3)$$

We also assume that for all $i \in \llbracket 1, m \rrbracket$, the function f_i satisfies $f_i(0) = 0$ and that system (1.2) is cooperative, which means:

$$f_i \in C^1(\mathbb{R}^m) \quad \text{and} \quad \partial_j f_i > 0, \quad \text{on } \mathbb{R}^m, \quad \text{for all } j \in \llbracket 1, m \rrbracket, \quad j \neq i. \quad (1.4)$$

We will make additional assumptions on the reaction term $F = (f_i)_{i=1}^m$ that are not general but enable us to understand the long time behaviour of a class of monotone systems.

The aim of this paper is to understand the time asymptotic location of the level sets of solutions to (1.2). We show that the speed of propagation is exponential in time, with a precise exponent depending on the smallest index $\alpha := \min_{i \in \llbracket 1, m \rrbracket} \alpha_i$ and on the principal eigenvalue of the matrix $DF(0)$ where $F = (f_i)_{i=1}^m$. Also we note that this speed does not depend on the space direction.

For what follows and without loss of generality, we suppose that $\alpha_{i+1} \leq \alpha_i$ for all $i \in \llbracket 1, m-1 \rrbracket$ so that $\alpha = \alpha_m < 1$. Before stating the main results, we need some additional hypotheses on the nonlinearities f_i , for all $i \in \llbracket 1, m \rrbracket$.

(H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is positive,

(H2) There exists $\Lambda > 1$ such that, for all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \geq \Lambda$, we have $f_i(s) \leq 0$,

(H3) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \geq c_{\delta_1} s_i^{1+\delta_1}$,

(H4) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \leq c_{\delta_2} |s|^{1+\delta_2}$,

(H5) $F = (f_i)_{i=1}^m$ is globally Lipschitz on \mathbb{R}^m ,

where the constants c_{δ_1} and c_{δ_2} are positive and independent of $i \in \llbracket 1, m \rrbracket$, and for all $j \in \{1, 2\}$

$$\delta_j \geq \frac{2}{d+2\alpha}.$$

This lower bound on δ_1 and δ_2 is a technical assumption to make the supersolution and subsolution to (1.2), we construct, to be regular enough. Note that one may easily produce examples of functions F satisfying (H1) to (H5).

We are now in a position to state our main theorem, which show that the solution to (1.2) move exponentially fast in time.

Theorem 1.1 *Let $d \geq 1$ and assume that F satisfies (1.4) and (H1) to (H5). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3). Then there exists $\tau > 0$ large enough such that for all $i \in \llbracket 1, m \rrbracket$, the following two facts are satisfied:*

a) For every $\mu_i > 0$, there exists a constant $c > 0$ such that,

$$u_i(t, x) < \mu_i, \quad \text{for all } t \geq \tau \text{ and } |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}.$$

b) There exist constants $\varepsilon_i > 0$ and $C > 0$ such that,

$$u_i(t, x) > \varepsilon_i, \quad \text{for all } t \geq \tau \text{ and } |x| < Ce^{\frac{\lambda_1}{d+2\alpha}t}.$$

The plan to set Theorem 1.1 is organized as follows. First, in the short section 2, we state a local existence result of solutions for cooperative systems involving fractional diffusion and we state a comparison principle for this type of solutions which, although standard, is crucial for the sequel. In Section 3 we deal with finite time and large x decay estimates. The end of this paper, Section 4 is devoted to the proof of Theorem 1.1.

2 Local existence and comparison principle

Recall that the operator $A = -\text{diag}((-\Delta)^{\alpha_1}, \dots, (-\Delta)^{\alpha_m})$ is sectorial (see [13]) in $(L^2(\mathbb{R}^d))^m$, with domain $D(A) = H^{2\alpha_1}(\mathbb{R}^d) \times \dots \times H^{2\alpha_m}(\mathbb{R}^d)$. If now u_0 satisfies the assumptions of Theorem 1.1, it is in $(L^2(\mathbb{R}^d))^m$, so that the Cauchy Problem (1.2) has a unique maximal solution, defined on an interval of the form $[0, t_{\max})$; moreover the L^2 -norm of u blows up as $t \rightarrow t_{\max}$ if $t_{\max} < +\infty$. Finally, we have $u \in C((0, t_{\max}), D(A)) \cap C([0, t_{\max}), (L^2(\mathbb{R}^d))^m)$ and $\frac{du}{dt} \in C((0, t_{\max}), (L^2(\mathbb{R}^d))^m)$. A standard iteration argument and Sobolev embeddings then yield

$$u \in C^p((0, t_{\max}), (H^q(\mathbb{R}^d))^m)$$

for every integer p and q .

Theorem 2.1 *Consider $T > 0$, and let $u = (u_i)_{i=1}^m$ and $v = (v_i)_{i=1}^m$ such that: $u \in C((0, T], D(A)) \cap C([0, T], (L^2(\mathbb{R}^d))^m) \cap C^1((0, T), (L^2(\mathbb{R}^d))^m)$; and $v \in C([0, T] \times \mathbb{R}^d) \cap C^1((0, T) \times \mathbb{R}^d)$. Assume that, for all $i \in \llbracket 1, m \rrbracket$, we have*

$$\partial_t u_i + (-\Delta)^{\alpha_i} u_i \leq f_i(u), \quad \partial_t v_i + (-\Delta)^{\alpha_i} v_i \geq f_i(v),$$

where f_i satisfies (1.4). If for all $i \in \llbracket 1, m \rrbracket$ and $x \in \mathbb{R}^d$, $u_i(0, x) \leq v_i(0, x)$ we have

$$u(t, x) \leq v(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^d.$$

Proof. Let us define for all $i \in \llbracket 1, m \rrbracket$, $w_i = u_i - v_i$. Then w_i satisfies $w_i(0, x) \leq 0$ and

$$\begin{aligned} \partial_t w_i + (-\Delta)^{\alpha_i} w_i &\leq f_i(u) - f_i(v) = \int_0^1 \nabla f_i(\sigma u + (1 - \sigma)v) d\sigma \cdot (u - v) \\ &= \int_0^1 \nabla f_i(\zeta_\sigma) d\sigma \cdot w, \end{aligned} \tag{2.1}$$

where $\zeta_\sigma = \sigma u + (1 - \sigma)v$. Notice now that $w_i^+ \in C((0, T), H^{2\alpha_i}(\mathbb{R}^d)) \cup W^{1,\infty}((0, T), L^2(\mathbb{R}^d))$. So, taking the scalar product of (2.1) with the vector function $(w_i^+)_{i=1}^m$ and integrating over \mathbb{R}^d , we have

$$\int_{\mathbb{R}^d} w_i^+ \partial_t w_i dx + \int_{\mathbb{R}^d} w_i^+ (-\Delta)^{\alpha_i} w_i dx \leq \int_{\mathbb{R}^d} w_i^+ \int_0^1 \nabla f_i(\zeta_\sigma) d\sigma \cdot w dx. \quad (2.2)$$

Recall that $\int_{\mathbb{R}^d} w_i^+ (-\Delta)^{\alpha_i} w_i dx \geq 0$. So we have, since $\partial_j f_i(\zeta_\sigma) \geq 0$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[\int_{\mathbb{R}^d} (w_i^+)^2 dx \right] &\leq \int_{\mathbb{R}^d} \int_0^1 \partial_i f_i(\zeta_\sigma) d\sigma (w_i^+)^2 dx + \sum_{j=1, j \neq i}^m \int_{\mathbb{R}^d} \int_0^1 \partial_j f_i(\zeta_\sigma) d\sigma w_i^+ w_j^+ dx \\ &\leq C \sum_{j=1}^m \int_{\mathbb{R}^d} (w_j^+)^2 dx, \end{aligned}$$

where C is a constant that depends on m . Doing this procedure for each $i \in \llbracket 1, m \rrbracket$ and adding, we get for $t \in [0, T]$

$$\frac{d}{dt} \left[\sum_{i=1}^m \int_{\mathbb{R}^d} (w_i^+)^2 dx \right] \leq C \sum_{i=1}^m \int_{\mathbb{R}^d} (w_i^+)^2 dx.$$

So, by Gronwall's inequality, we have $w_i \leq 0$ in $[0, T] \times \mathbb{R}^d$. \square

3 Finite time bounds and global existence

From hypothesis (H2), we deduce that the positive vector $M = \Lambda \mathbf{1}$, where $\mathbf{1}$ is the vector of size m with all entries equal to 1, is a supersolution to (1.2), if the initial condition $u_0 = (u_{0i})_{i=1}^m$ is smaller than M . So, from Theorem 2.1, we have $0 \leq u(t, x) \leq M$. To prove global existence, it remains to prove a locally finite L^2 bound; this is done in the next subsection.

3.1 Upper bound

Now, we are in position to establish an algebraic upper bound for the solutions of (1.2). From (H5), we know that, for $i \in \llbracket 1, m \rrbracket$ and $j \in \llbracket 1, m \rrbracket$

$$|\partial_j f_i(s)| \leq \text{Lip}(f_i), \quad \text{for all } s \in \mathbb{R}^m,$$

where $\text{Lip}(f_i)$ is the Lipschitz constant of f_i . Taking $l = \max_{i \in \llbracket 1, m \rrbracket} \text{Lip}(f_i)$, we have for all $s = (s_i)_{i=1}^m \geq 0$

$$f_i(s) = \int_0^1 Df_i(\sigma s) d\sigma \cdot s \leq \left| \sum_{j=1}^m s_j \int_0^1 \frac{\partial f_i}{\partial s_j}(\sigma s) d\sigma \right| \leq l \sum_{j=1}^m s_j. \quad (3.1)$$

Let us consider $v = (v_i)_{i=1}^m$ the solution of the following system

$$\begin{cases} \partial_t v + Lv &= Bv, \quad t > 0, x \in \mathbb{R}^m \\ v(0, \cdot) &= u_0, \quad \mathbb{R}^m, \end{cases} \quad (3.2)$$

where $L = \text{diag}((-\Delta)^{\alpha_1}, \dots, (-\Delta)^{\alpha_m})$, $B = (b_{ij})_{i,j=1}^m$ is a matrix with $b_{ij} = l$ for all $i, j \in \llbracket 1, m \rrbracket$. By (3.1) and Theorem 2.1, we conclude that $u \leq v$ in $[0, +\infty) \times \mathbb{R}^d$. A finite time upper bound for u is given by the following lemma.

Lemma 3.1 *Let $d \geq 1$ and let $u = (u_i)_{i=1}^m$ be the solution of system (1.2), with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3), and reaction term $F = (f_i)_{i=1}^m$ satisfying (1.4) and (H1) to (H5). Then, for all $i \in \llbracket 1, m \rrbracket$, there exists a locally bounded functions $C_1 : (0, +\infty) \rightarrow \mathbb{R}_+$ such that for all $t > 0$ and $|x|$ large enough, we have*

$$u_i(t, x) \leq \frac{C_1(t)}{|x|^{d+2\alpha}}.$$

Taking Fourier transforms in each term of system (3.2), we have

$$\begin{cases} \partial_t \mathfrak{F}(v) &= (A(|\xi|) + B)\mathfrak{F}(v), \quad \xi \in \mathbb{R}^d, t > 0 \\ \mathfrak{F}(v)(0, \cdot) &= \mathfrak{F}(u_0), \quad \mathbb{R}^d, \end{cases}$$

where $A(|\xi|) = \text{diag}(-|\xi|^{2\alpha_1}, \dots, -|\xi|^{2\alpha_m})$. Thus, we have that

$$\mathfrak{F}(v)(t, \xi) = e^{(A(|\cdot|) + B)t} \mathfrak{F}(u_0)(\xi)$$

and then, for all $x \in \mathbb{R}^d$ and $t \geq 0$:

$$u(t, x) \leq v(t, x) = \mathfrak{F}^{-1}(e^{(A(|\cdot|) + B)t} * u_0(x)). \quad (3.3)$$

In what follows, we prove that for each time $t > 0$, the solution of (1.2) decays as $|x|^{-d-2\alpha}$ for large values of $|x|$. Due to the decay of u_0 at infinity, we only need to prove that the entries of $\mathfrak{F}^{-1}(e^{(A(|\cdot|) + B)t})$ have the desired decay. The following lemma is needed to prove that we can rotate the integration line of a small angle $\varepsilon > 0$ in the expression of $\mathfrak{F}^{-1}(e^{(A(|\cdot|) + B)t})$.

Lemma 3.2 *For all $z \in \{z \in \mathbb{C} \mid 0 \leq \arg(z) < \frac{\pi}{4\alpha_1}\}$ and $t \geq 0$, we have*

$$\left\| e^{(A(z) + B)t} \right\| \leq e^{(\|B\| - |z|^{2\alpha_1} \cos(2\alpha_1 \arg(z)))t} + e^{(\|B\| - |z|^{2\alpha} \cos(2\alpha_1 \arg(z)))t}, \quad (3.4)$$

and if

$$I_t(z) := \int_0^t e^{(t-s)(A(z) + B)} [e^{sB}, A(z)] e^{sA(z)} ds, \quad (3.5)$$

then there exists $C_2 : (0, \infty) \rightarrow \mathbb{R}_+$ a locally bounded function such that

$$\|I_t(z)\| \leq C_2(t) (|z|^{2\alpha} e^{-|z|^{2\alpha} \cos(2\alpha_1 \arg(z))t} + |z|^{2\alpha_1} e^{-|z|^{2\alpha_1} \cos(2\alpha_1 \arg(z))t}). \quad (3.6)$$

Proof. Let z be in $\{z \in \mathbb{C} \mid 0 \leq \arg(z) < \frac{\pi}{4\alpha_1}\}$. There exist $j \in \llbracket 1, m \rrbracket$, and $k \in \llbracket 1, m \rrbracket$ such that $\|e^{(A(z)+B)t}\| = (e^{(A(z)+B)t})_{jk}$. Consider the system

$$\begin{cases} \partial_t w &= (A(z) + B)w, & z \in \mathbb{C}, t > 0, \\ w(0, z) &= e_j & z \in \mathbb{C}, \end{cases}$$

where e_j is the j th vector of the canonical basis of \mathbb{R}^m . Thus, we have

$$w(t, z) = e^{(A(z)+B)t} \cdot e_j \quad \text{and} \quad \|w\| = \|e^{(A(z)+B)t}\|.$$

Multiply (3.1) by the conjugate transpose \overline{w} and take the real part to get

$$\frac{1}{2} \partial_t \|w\|^2 + \sum_{l=1}^m \cos(2\alpha_l \arg(z)) |z|^{2\alpha_l} |w_l|^2 = \operatorname{Re}(Bw \cdot \overline{w}) \leq \|B\| \|w\|^2.$$

The choice of $\arg(z)$ and Gronwall's Lemma end the proof.

To prove (3.6), it is sufficient to notice that, for $s \in [0, t]$, we have

$$\begin{aligned} \left\| e^{sA(|z|e^{i\arg(z)})} \right\| &\leq e^{-|z|^{2\alpha_1} \cos(2\alpha_1 \arg(z))s} + e^{-|z|^{2\alpha_1} \cos(2\alpha_1 \arg(z))s}, \\ \left\| [e^{sB}, A(|z|e^{i\arg(z)})] \right\| &\leq C(t)(|z|^{2\alpha} + |z|^{2\alpha_1}), \end{aligned}$$

where $C : (0, +\infty) \rightarrow \mathbb{R}_+$ is a locally bounded function, and due to (3.4), we also have

$$\left\| e^{(A(|z|e^{i\arg(z)})+B)(t-s)} \right\| \leq e^{(\|B\| - |z|^{2\alpha_1} \cos(2\alpha_1 \arg(z)))(t-s)} + e^{(\|B\| - |z|^{2\alpha} \cos(2\alpha_1 \arg(z)))(t-s)}.$$

□

Proof for $d = 1$. In this proof, we denote by $C : (0, +\infty) \rightarrow \mathbb{R}_+$ a locally bounded function. From (3.3), we only have to find an upper bound to $\mathfrak{F}^{-1}(e^{(A(\cdot)+B)t})$. First, we consider for $t \geq 0$ and $z \in \mathbb{C}$, $w(t, z) := e^{tB} e^{tA(z)}$. Thus, w satisfies the Cauchy problem

$$\begin{cases} \partial_t w &= (A(z) + B)w + [e^{tB}, A(z)]e^{tA(z)}, & t > 0, z \in \mathbb{C} \\ w(0, z) &= Id, & z \in \mathbb{C}, \end{cases}$$

where $[e^{tB}, A(z)] = e^{tB}A(z) - A(z)e^{tB}$. By Duhamel's formula, we get for all $z \in \mathbb{C}$ and $t \geq 0$:

$$e^{t(A(z)+B)} = e^{tB} e^{tA(z)} - \int_0^t e^{(t-s)(A(z)+B)} [e^{sB}, A(z)] e^{sA(z)} ds. \quad (3.7)$$

Thus, for all $t > 0$ and all $x \in \mathbb{R}$, we have

$$\begin{aligned} \mathfrak{F}^{-1}(e^{(A(\cdot)+B)t})(x) &= \int_{\mathbb{R}} e^{ix\xi} e^{(A(|\xi|)+B)t} d\xi \\ &= \int_{\mathbb{R}} e^{ix\xi} e^{tB} e^{tA(|\xi|)} d\xi - \int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi \\ &= e^{tB} \operatorname{diag}(p_{\alpha_1}(t, x), \dots, p_{\alpha_m}(t, x)) - \int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi, \end{aligned} \quad (3.8)$$

where for $i \in \llbracket 1, m \rrbracket$, p_{α_i} is the heat kernel of the operator $(-\Delta)^{\alpha_i}$ in \mathbb{R} , [3]. Since $\alpha = \min_{i \in \llbracket 1, m \rrbracket} \alpha_i \in (0, 1)$, for large values of $|x|$, we clearly have

$$\|e^{tB} \text{diag}(p_{\alpha_1}(t, x), \dots, p_{\alpha_m}(t, x))\| \leq \frac{C(t)}{|x|^{1+2\alpha}}. \quad (3.9)$$

It remains to bound from above the following quantity:

$$\int_{\mathbb{R}} e^{ix\xi} I_t(|\xi|) d\xi = 2 \int_0^\infty \cos(xr) I_t(r) dr = 2 \Re \left(\int_0^\infty e^{ixr} I_t(r) dr \right).$$

We use the following two facts. First, for all $t \geq 0$, the function $z \mapsto e^{ixz} I_t(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$. Second, for $\delta > 0$ (respectively $R > 0$), on the arc $\{\pm \delta e^{i\theta}, \theta \in [0, \varepsilon]\}$ (respectively $\{\pm R e^{i\theta}, \theta \in [0, \varepsilon]\}$), the entries of I_t tends to 0 as δ tends to 0 (respectively R tends to $+\infty$, due to Lemma 3.2). Consequently, we can rotate the integration line of a small angle $\varepsilon \in (0, \frac{\pi}{4\alpha_1})$ and the quantity we have to bound from above becomes $\int_0^\infty e^{ixre^{i\varepsilon}} I_t(re^{i\varepsilon}) dr$, with

$$I_t(re^{i\varepsilon}) = \int_0^t e^{(t-s)(A(re^{i\varepsilon})+B)} [e^{sB}, A(re^{i\varepsilon})] e^{sA(re^{i\varepsilon})} ds.$$

From Lemma 3.2, taking

$$\eta_t = \left\| \int_0^\infty e^{ixre^{i\varepsilon}} I_t(re^{i\varepsilon}) dr \right\|$$

we get, for large values of $|x|$

$$\begin{aligned} \eta_t &\leq C(t) \int_0^\infty e^{-xr \sin(\varepsilon)} (r^{2\alpha} e^{-r^{2\alpha} \cos(2\alpha_1 \varepsilon) t} + r^{2\alpha_1} e^{-r^{2\alpha_1} \cos(2\alpha_1 \varepsilon) t}) dr \\ &\leq \frac{C(t)}{|x|^{1+2\alpha}} \int_0^\infty e^{-\tilde{r} \sin(\varepsilon)} (\tilde{r}^{2\alpha} e^{-\frac{\tilde{r}^{2\alpha}}{|x|^{2\alpha}} \cos(2\alpha_1 \varepsilon) t} + \tilde{r}^{2\alpha_1} e^{-\frac{\tilde{r}^{2\alpha_1}}{|x|^{2\alpha_1}} \cos(2\alpha_1 \varepsilon) t}) d\tilde{r} \\ &\leq \frac{C(t)}{|x|^{1+2\alpha}}. \end{aligned} \quad (3.10)$$

With (3.8), (3.9) and (3.10), we conclude that for large values of $|x|$ and for all $t \geq 0$

$$\left\| \mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})(x) \right\| \leq \frac{C(t)}{|x|^{1+2\alpha}},$$

which concludes the proof. \square

Now, we state the proof of Lemma 3.1 in the higher space dimension case, i.e. when $d > 1$.

Proof. As previously, from (3.3), we only need to bound from above the function $\mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})$. Let $t > 0$ and $|x| > 1$, using the spherical coordinates system in dimension $d > 1$, the definition of Bessel Function of first kind (see [14] and [15]), we have

$$\begin{aligned} \mathfrak{F}^{-1}(e^{(A(|\cdot|)+B)t})(x) &= C_d \int_0^\infty \int_{-1}^1 e^{(A(r)+B)t} \cos(|x|rs) r^{d-1} (1-s^2)^{\frac{d-3}{2}} ds dr \\ &= \frac{C_d}{|x|^{\frac{d}{2}-1}} \int_0^\infty e^{(A(r)+B)t} J_{\frac{d}{2}-1}(|x|r) r^{\frac{d}{2}} dr, \end{aligned}$$

where C_d is a positive constant depending on d .

The matrix $e^{(A(r)+B)t}$ is split into two pieces as done in (3.7), to get

$$\mathfrak{F}^{-1}(e^{(A(\cdot)+B)t})(x) = e^{tB} \text{diag}(p_{\alpha_1}(t, x), \dots, p_{\alpha_m}(t, x)) - \frac{C_d}{|x|^{\frac{d}{2}-1}} \int_0^\infty I_t(r) J_{\frac{d}{2}-1}(|x|r) r^{\frac{d}{2}} dr,$$

where I_t has been defined in (3.5). From [3], the first piece of the right hand side has the correct algebraic decay. It remains to bound from above the second piece. In fact, using the Whittaker function (defined in [15] for example), we have for all $x \in \mathbb{R}^d$ and all $t > 0$:

$$\begin{aligned} \frac{C_d}{|x|^{\frac{d}{2}-1}} \int_0^\infty I_t(r) J_{\frac{d}{2}-1}(|x|r) r^{\frac{d}{2}} dr &= \frac{C_d}{|x|^{\frac{d-1}{2}} \sqrt{2\pi}} \Re \left(\int_0^\infty I_t(r) e^{\frac{d-1}{4}i\pi} W_{0, \frac{d}{2}-1}(2i|x|r) r^{\frac{d-1}{2}} dr \right) \\ &= \frac{C_d}{|x|^d \sqrt{2\pi}} \Re \left(\int_0^\infty I_t(\tilde{r}|x|^{-1}) e^{\frac{d-1}{4}i\pi} W_{0, \frac{d}{2}-1}(2i\tilde{r}) \tilde{r}^{\frac{d-1}{2}} d\tilde{r} \right) \end{aligned}$$

As done in the one dimension case, since the Whittaker function is bounded, we can rotate the integration line of a small angle $\varepsilon \in (0, \frac{\pi}{4\alpha_1})$. Thus, using (3.6), we have the result if we prove that the following integral

$$\int_0^\infty \left| W_{0, \frac{d}{2}-1}(2i\tilde{r}e^{i\varepsilon}) \right| \tilde{r}^{\frac{d-1}{2}} (\tilde{r}^{2\alpha} + \tilde{r}^{2\alpha_1}) d\tilde{r}$$

is convergent. From [14], $W_{0, \frac{d}{2}-1}$ has the following asymptotic expressions, thus $W_{0, \frac{d}{2}-1}(z) \underset{|z| \rightarrow +\infty}{\sim} e^{-\frac{z}{2}}$ and

$$W_{0, \frac{d}{2}-1}(z) \underset{|z| \rightarrow 0}{\sim} \begin{cases} -\Gamma(\frac{d-1}{2})^{-1} \left(\ln(z) + \frac{\Gamma'(\frac{d-1}{2})}{\Gamma(\frac{d-1}{2})} \right) z^{\frac{d-1}{2}}, & \text{if } d = 2 \\ \frac{\Gamma(d-2)}{\Gamma(\frac{d-1}{2})} z^{\frac{3-d}{2}}, & \text{if } d \geq 3. \end{cases}$$

□

3.2 Lower bound

The following result is important and needed to prove Theorem 1.1. It sets an algebraically lower bound for the solutions of the cooperative system (1.2). This result is valid for any dimension $d \in \mathbb{N}^*$. Moreover, since for all $i \in \llbracket 1, m \rrbracket$, $f_i(0) = 0$, we have for all $s = (s_i)_{i=1}^m \in \mathbb{R}^m$ with $0 \leq s \leq M$

$$f_i(s) = \int_0^1 Df_i(\sigma s) d\sigma \cdot s = \sum_{j=1}^m s_j \int_0^1 \frac{\partial f_i}{\partial s_j}(\zeta_\sigma) d\sigma$$

where $\zeta_\sigma = \sigma s \in [0, M]$ and $\frac{\partial f_i}{\partial s_j} : [0, M] \rightarrow \mathbb{R}$ is continuous for all $i, j \in \llbracket 1, m \rrbracket$, since the system is cooperative, there exist constants $\gamma_{ij} > 0$ such that for all $i \in \llbracket 1, m \rrbracket$ and $j \in \llbracket 1, m \rrbracket$:

$$|\partial_i f_i(\zeta_\sigma)| \leq \gamma_{ii} \quad \text{and} \quad \gamma_{ij} \leq \partial_j f_i(\zeta_\sigma). \quad (3.1)$$

Lemma 3.3 *Let $u = (u_i)_{i=1}^m$ be the solution of the system (1.2), with non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3) and with reaction term $F = (f_i)_{i=1}^m$ satisfying (1.4), (H1), (H2) and (H5). Then, for all $i \in \llbracket 1, m \rrbracket$, $x \in \mathbb{R}^d$ and $t \geq 1$, we have:*

$$u_i(t, x) \geq \frac{\underline{c} t e^{-\gamma_{mm} t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \quad (3.2)$$

where \underline{c} is a positive constant and γ_{mm} is defined in (3.1).

Proof. We split the proof into three steps: first, we prove the result for $i = m$, which serves as an initiation of the process. In an intermediate step, for all $i \in \llbracket 1, m-1 \rrbracket$, $t \geq 1$ and $s \in [0, t-1]$, we find a lower bound of $p_{\alpha_i}(\cdot, t-s) \star (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1}$, that decays like $|x|^{-(d+2\alpha)}$ for large values of $|x|$. In a third step, for all $i \in \llbracket 1, m-1 \rrbracket$, $t \geq 1$ and $s \in [0, t-1]$, we prove that $u_i(t, \cdot)$ can be bounded from below by an expression that only depends on the integral $\int_0^t p_{\alpha_i}(\cdot, t-s) \star (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1} ds$.

Step 1. We have for all $x \in \mathbb{R}^d$ and $t > 0$:

$$\partial_t u_m + (-\Delta)^{\alpha_m} u_m = f_m(u) \geq \int_0^1 \partial_m f_m(\zeta_\sigma) d\sigma u_m \geq -\gamma_{mm} u_m,$$

where γ_{mm} is defined in (3.1). By the maximum principle of reaction diffusion equations, we have for all $t \geq 0$

$$u_m(t, x) \geq e^{-\gamma_{mm} t} (p_{\alpha_m}(t, \cdot) * u_{0m})(x),$$

Since $u_{0m}(\cdot) \not\equiv 0$ is continuous and nonnegative, we can find $\xi \in \mathbb{R}^d$ such that $u_{0m}(y) \geq C$ for all $y \in B_R(\xi)$ for some $R > 0$ and $C > 0$. If $|x| > R$, $t \geq 1$ and using that $\alpha := \alpha_m < 1$, we get

$$\begin{aligned} (p_{\alpha_m}(t, \cdot) * u_{0m})(x) &\geq C \int_{|y-\xi| \leq R} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy \\ &= C \int_{|z| \leq R} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + |x-\xi-z|^{d+2\alpha}} dz. \end{aligned}$$

We also have $|x-\xi-z| \leq (2 + \frac{\xi}{R})|x|$. Thus

$$t^{\frac{d}{2\alpha}+1} + |x-\xi-z|^{d+2\alpha} \leq \left(2 + \frac{\xi}{R}\right)^{d+2\alpha} t^{\frac{d}{2\alpha}+1} + \left(2 + \frac{\xi}{R}\right)^{d+2\alpha} |x|^{d+2\alpha}.$$

Then

$$(p_{\alpha_m}(t, \cdot) * u_{0m})(x) \geq \frac{CB^{-1}}{(2 + \frac{\xi}{R})^{d+2\alpha}} \int_{|z| \leq R} \frac{t}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} dz = \frac{\tilde{C}t}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}},$$

where \tilde{C} is a positive constant. If $|x| \leq R$ and $t \geq 1$,

$$\begin{aligned} (p_{\alpha_m}(t, \cdot) * u_{0m})(x) &\geq \int_{B_1(0)} \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} u_{0m}(y) dy \\ &\geq \frac{B^{-1}t}{t^{\frac{d}{2\alpha}+1} + (R+1)^{d+2\alpha}} \int_{B_1(0)} u_{0m}(y) dy \\ &\geq \frac{\bar{C}t}{t^{\frac{d}{2\alpha}+1}} \geq \frac{\bar{C}t}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}, \end{aligned}$$

for some small constant $\bar{C} > 0$. Then, there exist $C_m > 0$ such that for all $x \in \mathbb{R}^d$ and $t \geq 1$

$$u_m(t, x) \geq \frac{C_m t e^{-\gamma_{mm}t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}. \quad (3.3)$$

Step 2. By similar computations as done in Step 1, it is possible to find a constant $C > 0$ such that for all $x \in \mathbb{R}^d$, $t \geq 1$ and $s \in [0, t-1]$:

- if $\alpha_i = 1$ then

$$\begin{aligned} p_{\alpha_i}(\cdot, t-s) \star (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1}(x) &\geq \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{4(t-s)}}}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy \\ &\geq \frac{1}{(4\pi(t-s))^{\frac{d}{2}} (s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha})}, \end{aligned}$$

- if $\alpha_i \in (0, 1)$ then

$$\begin{aligned} p_{\alpha_i}(\cdot, t-s) \star (s^{\frac{d}{2\alpha}+1} + |\cdot|^{d+2\alpha})^{-1}(x) &\geq \int_{\mathbb{R}^d} \frac{1}{((t-s)^{\frac{d}{2\alpha_i}+1} + |y|^{d+2\alpha_i})(s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha})} dy \\ &\geq \frac{(t-s)^{-\frac{d}{2\alpha_i}}}{s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}. \end{aligned}$$

Step 3. For $i \in \llbracket 1, m-1 \rrbracket$, we have for all $x \in \mathbb{R}^d$ and $t \geq 0$

$$\partial_t u_i + (-\Delta)^{\alpha_i} u_i \geq \int_0^1 \partial_m f_i(\zeta_\sigma) d\sigma u_m + \int_0^1 \partial_i f_i(\zeta_\sigma) d\sigma u_i \geq \gamma_{im} u_m - \delta_i u_i,$$

where $\zeta_\sigma = \sigma u$ and $\delta_i \geq \max(\gamma_{ii}, \gamma_m + 1)$. Then, by the maximum principle of reaction diffusion equations and Duhamel's formula, we have for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$u_i(t, x) \geq e^{-\delta_i t} (p_{\alpha_i}(t, \cdot) * u_{0i})(x) + \gamma_{im} e^{-\delta_i t} \int_0^t \int_{\mathbb{R}^d} p_{\alpha_i}(t-s, y) u_{0m}(s, x-y) e^{\delta_i s} dy ds.$$

So, taking $t \geq 1$, and using (3.3), we get

$$u_i(t, x) \geq C_m \gamma_{im} e^{-\delta_i t} \int_0^{t-1} \int_{\mathbb{R}^d} p_{\alpha_i}(t-s, y) \frac{s e^{(\delta_i - \gamma_{mm})s}}{s^{\frac{d}{2\alpha}+1} + |x-y|^{d+2\alpha}} dy ds$$

Using Step 2, we get the following lower bound, for all $x \in \mathbb{R}^d$, $t \geq 1$, taking C_i smaller if necessary:

$$u_i(t, x) \geq C_i \frac{e^{-\delta_i t}}{t^{\frac{d}{2\alpha}}} \int_0^{t-1} \frac{s e^{(\delta_i - \gamma_{mm})s} - 1}{s^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}} ds \geq \frac{C_i t e^{-\gamma_{mm} t}}{t^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}.$$

□

4 Proof of Theorem 1.1

Inspired by the formal analysis done in [4], we construct an explicit supersolution (respectively subsolution) of the form

$$v(t, x) = a \left(1 + b(t) |x|^{\delta(d+2\alpha)} \right)^{-\frac{1}{\delta}} \phi_1, \quad (4.1)$$

where $b(t)$ is a time continuous function asymptotically proportional to $e^{-\delta \lambda_1 t}$, $\phi_1 = (\phi_{1,i})_{i=1}^m \in \mathbb{R}^m$ is the normalised (positive) principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1 , and δ is equal to δ_1 (respectively δ_2) defined in (H3) (respectively (H4)).

Lemma 4.1 *Let v be defined as in (4.1). Then, there exist a constant $D > 0$ such that for all $i \in \llbracket 1, m \rrbracket$, $t > 0$ and $x \in \mathbb{R}^d$*

$$|(-\Delta)^{\alpha_i} v_i(t, x)| \leq D b(t)^{\frac{2\alpha_i}{\delta(d+2\alpha)}} v_i(t, x),$$

where $\alpha_i \in (0, 1]$.

Proof. The case $\alpha_i = 1$ is trivial. For $\alpha_i \in (0, 1)$ and $\delta \geq \frac{2}{d+2\alpha}$, since $(-\Delta)^{\alpha_i}$ is $2\alpha_i$ -homogeneous, we only need to prove

$$|(-\Delta)^{\alpha_i} w(x)| \leq D w(x)$$

where $w(x) = (1 + |x|^{\delta(d+2\alpha)})^{-\frac{1}{\delta}}$.

We consider the following decomposition, which is the central part of the proof:

$$\begin{aligned} (-\Delta)^{\alpha_i} w(x) &= \int_{|y| > 3|x|/2} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy + \int_{B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy \\ &\quad + \int_{\{|x| \leq 2|y| \leq 3|x|\} \setminus B_{|x|/2}(x)} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy + \int_{|y| \leq |x|/2} \frac{w(x) - w(y)}{|x-y|^{d+2\alpha_i}} dy. \end{aligned}$$

Each piece is easily bounded, as in [16] for instance. \square

In what follows, we will use the results of previous sections to obtain appropriate sub and super solutions to (1.2) of the form (4.1). We divide the proof of Theorem 1.1 in two lemmas.

Lemma 4.2 *Assume that F satisfies (1.4), (H1), (H2), (H3) and (H5). Let u be the solution to (1.2) with u_0 satisfying the assumptions of Theorem 1.1. Then, for every $\mu = (\mu_i)_{i=1}^m > 0$, there exists $c > 0$ such that, for all $t > \tau$, with $\tau > 0$ large enough*

$$\left\{x \in \mathbb{R}^d \mid |x| > ce^{\frac{\lambda_1}{d+2\alpha}t}\right\} \subset \left\{x \in \mathbb{R}^d \mid u(t, x) < \mu\right\}.$$

Proof: We consider the function \bar{u} given by (4.1) with $\delta = \delta_1$ as in (H3). The idea is to adjust $a > 0$ and $b(t)$ so that the function \bar{u} serves as supersolution of (1.2).

In the sequel, a is any positive constant satisfying

$$a \geq \left(\frac{D + \lambda_1}{c_h}\right)^{\frac{1}{\delta_1}} \max_{i \in \llbracket 1, m \rrbracket} \left(\frac{1}{\phi_{1,i}}\right),$$

where c_h is defined in (H2). For any constant $B \in (0, (1 + D\lambda_1^{-1})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}})$, where $D > 0$ is given in Lemma 4.1, we consider the following ordinary differential equation

$$b'(t) + \delta_1 D b(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} + \delta_1 \lambda_1 b(t) = 0, \quad b(0) = (-D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_1(d+2\alpha)}})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}} \quad (4.2)$$

whose solution is given by

$$b(t) = (-D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_1(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t})^{-\frac{\delta_1(d+2\alpha)}{2\alpha}}.$$

For all $t \geq 0$, we have $0 \leq b(t) \leq b(0) \leq 1$. Using Lemma 4.1, we have for all $i \in \llbracket 1, m \rrbracket$

$$\begin{aligned} \partial_t \bar{u}_i + (-\Delta)^{\alpha_i} \bar{u}_i - f_i(\bar{u}) &= \partial_t \bar{u}_i + (-\Delta)^{\alpha_i} \bar{u}_i - D f_i(0) \bar{u} + [D f_i(0) \bar{u} - f_i(\bar{u})] \\ &\geq \frac{a \phi_{1,i}}{\delta_1 (1 + b(t) |x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}} \left\{ -b'(t) - \delta_1 D b(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}+1} - \delta_1 \lambda_1 b(t) \right\} |x|^{\delta_1(d+2\alpha)} \\ &\quad + \frac{a \phi_{1,i}}{(1 + b(t) |x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}+1}} \left\{ -D b(t)^{\frac{2\alpha}{\delta_1(d+2\alpha)}} - \lambda_1 + c_h \phi_{1,i}^{\delta_1} a^{\delta_1} \right\} \geq 0. \end{aligned}$$

Finally, due to Lemma 3.1, for a fixed $t_0 > 0$, there exists $t_1 \geq 0$ such that for all $x \in \mathbb{R}^d$ and all $i \in \llbracket 1, m \rrbracket$, we have $\bar{u}_i(t_1, x) \geq u_i(t_0, x)$. Thus, by Theorem 2.1 we have, for all $t \geq t_0$, all $x \in \mathbb{R}^d$ and all $i \in \llbracket 1, m \rrbracket$: $\bar{u}_i(t + t_1 - t_0, x) \geq u_i(t, x)$.

For any $(\mu_i)_{i=1}^m > 0$, we define for $i \in \llbracket 1, m \rrbracket$ the constants

$$c_i^{d+2\alpha} := a \phi_{1,i} e^{\lambda_1(t_1-t_0)} [\mu_i B^{\frac{1}{\delta_1}}]^{-1}.$$

Taking $c = \max_{i \in \llbracket 1, m \rrbracket} c_i$, if $|x| > ce^{\frac{\lambda_1}{d+2\alpha}t}$, then, for all $t > \tau := t_0$ and all $i \in \llbracket 1, m \rrbracket$

$$u_i(t, x) \leq \bar{u}_i(t + t_1 - t_0, x) = \frac{a\phi_{1,i}}{(1 + b(t + t_1 - t_0)|x|^{\delta_1(d+2\alpha)})^{\frac{1}{\delta_1}}} < \mu_i.$$

□

Lemma 4.3 *Let $d \geq 1$ and assume that F satisfies (1.4), (H1), (H2), (H4) and (H5). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3). Then, for all $i \in \llbracket 1, m \rrbracket$, there exist constants $\varepsilon_i > 0$ and $C > 0$ such that,*

$$u_i(t, x) > \varepsilon_i, \quad \text{for all } t \geq t_1 \text{ and } |x| < Ce^{\frac{\lambda_1}{d+2\alpha}t},$$

with $t_1 > 0$ large enough.

Proof: As in the previous proof, we consider the function \underline{u} given by (4.1) with $\delta = \delta_2$ defined in (H4). Since, $\underline{u}_i(0, \cdot) \leq u_{0i}$ may not hold for all $i \in \llbracket 1, m \rrbracket$, we look for a time $t_1 > 0$ such that $\underline{u}_i(0, \cdot) \leq u_i(t_1, \cdot)$ for all $i \in \llbracket 1, m \rrbracket$. Indeed, let L be a constant greater than $\max\{D, \lambda_1\}$, where D is given by Lemma 4.1. We choose $t_1 \geq \max(1, 2D\lambda_1^{-1})$ large enough, so that if we set

$$a = \frac{\min_{i \in \llbracket 1, m \rrbracket} C_i e^{-\gamma_{mm}t_1}}{2 \max_{i \in \llbracket 1, m \rrbracket} \phi_{1,i} t_1^{\frac{d}{2\alpha}}} \quad \text{and} \quad B = \left(\frac{2}{t_1}\right)^{\frac{(d+2\alpha)\delta_2}{2\alpha}}, \quad (4.3)$$

then

$$a \leq \left(\frac{\min_{i \in \llbracket 1, m \rrbracket} \phi_{1,i} \lambda_1}{2c_{\delta_2}}\right)^{\frac{1}{\delta_2}} \quad \text{and} \quad B \leq (D\lambda_1^{-1})^{-\frac{(d+2\alpha)}{2\alpha}\delta_2},$$

where c_{δ_2} is defined in (H4). Then we set

$$b(t) = (D\lambda_1^{-1} + B^{-\frac{2\alpha}{\delta_2(d+2\alpha)}} e^{\frac{2\alpha\lambda_1}{d+2\alpha}t})^{-\frac{(d+2\alpha)}{2\alpha}\delta_2}.$$

Using Lemma 4.1 and (H3), similarly to the previous proof, we can state that, for all $i \in \llbracket 1, m \rrbracket$,

$$\partial_t \underline{u}_i + (-\Delta)^{\alpha_i} \underline{u}_i - f_i(\underline{u}) \leq 0, \quad \text{in } (0, +\infty) \times \mathbb{R}^d.$$

From Lemma 3.3, we know that for all $i \in \llbracket 1, m \rrbracket$ and all $x \in \mathbb{R}^d$

$$u_i(t_1, x) \geq \frac{t_1 e^{-\gamma_{mm}t_1}}{t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}}.$$

By (4.3), we deduce

$$\underline{c}t_1 e^{-\gamma_{mm}t_1} (1 + b(0)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}} \geq \frac{\underline{c}}{2} t_1 e^{-\gamma_{mm}t_1} (1 + b(0)^{\frac{1}{\delta_2}} |x|^{d+2\alpha}) \geq a\phi_i(t_1^{\frac{d}{2\alpha}+1} + |x|^{d+2\alpha}).$$

Therefore, we get, for all $i \in \llbracket 1, m \rrbracket$, $u_i(t_1, \cdot) \geq \underline{u}_i(0, \cdot)$ in \mathbb{R}^d , and by Theorem 2.1, we have for all $t \geq t_1$

$$u_i(t, \cdot) \geq \underline{u}_i(t - t_1, \cdot), \quad \text{in } \mathbb{R}^d$$

Finally we choose

$$\varepsilon_i = \frac{a\phi_{1,i}}{2^{\frac{1}{\delta_2}}} \quad \text{and} \quad C^{d+2\alpha} = e^{-\lambda_1 t_1} B^{-\frac{1}{\delta_2}}.$$

If $t \geq \tau := t_1$ and $|x| \leq C e^{\frac{\lambda_1}{d+2\alpha}t}$, we have

$$u_i(t, x) \geq \underline{u}_i(t - t_1, x) = \frac{a\phi_{1,i}}{(1 + b(t - t_1)|x|^{\delta_2(d+2\alpha)})^{\frac{1}{\delta_2}}} \geq \frac{a\phi_{1,i}}{2^{\frac{1}{\delta_2}}} = \varepsilon_i.$$

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